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Chains of Differential Subvarieties in an Algebraic Variety

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Let X be a δ-variety over some δ-field $\mathcal{K}$. Denote by $\mathrm{td}_\delta(X/\mathcal{K})$, or simply $\mathrm{td}_\delta(X)$ if the ground field is understood, the δ-transcendental degree of $\mathcal{K}\langle X\rangle$ over $\mathcal{K}$. Suppose $\mathrm{td}_\delta(X) = d$; Johnson [*Comment. Math. Helv.* **44** (1969), 207–216] showed that there is an increasing chain of δ-subvarieties of length ωd in X. The question, also known as the Kolchin Catenary Problem, is: Given a point $x \in X$, is there an increasing chain of δ-subvarieties of length ωd starting at x? We will give an affirmative answer to this question if X is an algebraic variety.

1. PRELIMINARIES

We assume the readers are familiar with basic differential algebra. Our notations and terminology are standard and sometimes we use them without explicitly giving the definitions. However, readers can always find them in [Bu, Chapter 2] or [Ko, Chapter I].

A **δ-ring**[1] is a commutative ring with 1 containing $\mathbb{Q}$ equipped with a single derivation, denoted by δ. An ideal in a δ-ring is called a **δ-ideal** (or a differential ideal) if it is closed under the derivation. In a δ-ring, the radical of a δ-ideal is also a δ-ideal. Moreover, any radical δ-ideal is an intersection of prime δ-ideals. Therefore, by considering the nilradical, a δ-ring must process at least one δ-prime ideal. Note also that for a δ-ring $\mathcal{A}$ and δ-$\mathcal{A}$-algebras $\mathcal{B}$ and $\mathcal{C}$, if the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$ is not the zero ring, then it is easy to check that it contains $\mathbb{Q}$ and the natural derivation

[1]Here we are defining what is usually called a Ritt ring in the literature.

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defined by $\delta(b \otimes c) = \delta b \otimes c + b \otimes \delta c$ makes it into a δ-ring with the canonical maps from $\mathscr{B}$ and $\mathscr{C}$ into it being δ-homomorphisms.

Our starting point is the following result, which asserts that the classical going-up and going-down theorems still hold if we replace the set of prime ideals by the set of prime δ-ideals.

PROPOSITION 1.1. *Let $\mathscr{A} \subseteq \mathscr{B}$ be δ-integral domains with $\mathscr{B}$ integral over $\mathscr{A}$. Then the lying over theorem and the going-up theorem hold for the class of prime δ-ideals. Moreover, if $\mathscr{A}$ is integrally closed, then the going-down theorem holds for the class of prime δ-ideals also.*

Proof. Let $\mathfrak{p}$ be a prime δ-ideal in $\mathscr{A}$ and let $\kappa(\mathfrak{p})$ be its residue field. Since $\mathscr{B}$ is integral over $\mathscr{A}$, the usual lying over theorem implies $\mathscr{B} \otimes_{\mathscr{A}} \kappa(\mathfrak{p})$ has a prime ideal and hence is not the zero ring. So it is a δ-ring with the natural derivation, and hence processes a prime δ-ideal, say $\tilde{\mathfrak{q}}$. Moreover, since the canonical map from $\mathscr{B}$ into the tensor product is a δ-homomorphism, the contraction of $\tilde{\mathfrak{q}}$ in $\mathscr{B}$ is a prime δ-ideal and it is lying over $\mathfrak{p}$.

The going-up and going-down theorems for δ-ideals have similar proofs. Here we will only give the proof of the latter one. As usual, one can reduce to the case where $\mathfrak{p}_1$, $\mathfrak{p}_2$ are prime δ-ideals of $\mathscr{A}$, with $\mathfrak{p}_2 \subset \mathfrak{p}_1$ and where $\mathfrak{q}_1$ is a prime δ-ideal of $\mathscr{B}$ lying over $\mathfrak{p}_1$. Since $\mathscr{A}$ is integrally closed, by the usual going-down theorem, there is a prime ideal inside $\mathfrak{q}_1$ lying over $\mathfrak{p}_2$; therefore, the ring $\mathscr{S} = \mathscr{B}_{\mathfrak{q}_1} \otimes_{\mathscr{A}_{\mathfrak{p}_1}} k(\mathfrak{p}_2 \mathscr{A}_{\mathfrak{p}_1})$ is nonzero. As in the previous case, it has a natural δ-ring structure. Let $\mathfrak{q}_2 \subset \mathscr{B}$ be the preimage of a prime δ-ideal in $\mathscr{S}$. Then $\mathfrak{q}_2$ is a prime δ-ideal sitting inside $\mathfrak{q}_1$ and lying over $\mathfrak{p}_2$. ∎

The above proposition is strong enough for our subsequent arguments. However, there is one unsatisfactory point: it only guarantees that one of the prime ideals lying over a given prime δ-ideal is a δ-ideal. When is the case that all of them are δ-prime ideals? We will give a sufficient condition here. First, let us start with an easy lemma.

LEMMA 1.2. *Let $(\mathscr{L}, \delta_{\mathscr{L}})/(\mathscr{K}, \delta)$ be a separable algebraic extension of δ-fields. Let $(\mathscr{H}, \delta_{\mathscr{H}})$ be a δ-field extension of $(\mathscr{K}, \delta)$. Then every field homomorphism from $\mathscr{L}$ to $\mathscr{H}$ over $\mathscr{K}$ is a differential field homomorphism.*

Proof. Pick any $\varphi \in \mathrm{Hom}_{\mathscr{K}}(\mathscr{L}, \mathscr{H})$ and $x \in \mathscr{L}$. Let f be the minimal polynomial of x over $\mathscr{K}$. Then we have

$$0 = \delta_{\mathscr{L}}(0) = \delta_{\mathscr{L}}(f(x)) = f^{\delta}(x) + f'(x)\delta_{\mathscr{L}}(x),$$

where f^{δ} is the polynomial obtained by applying δ to the coefficients of f and f' is the formal derivative of f. Since x is separable over $\mathscr{K}$, we have

$f'(x) \neq 0$; hence,

$$\delta_{\mathscr{L}}(x) = -\frac{f^{\delta}(x)}{f'(x)}.$$

Note that f is also the minimal polynomial of $\varphi(x)$ over $\mathscr{K}$. By exactly the same argument, we have

$$\varphi(\delta_{\mathscr{L}}(x)) = \varphi\left(-\frac{f^{\delta}(x)}{f'(x)}\right) = -\frac{f^{\delta}(\varphi(x))}{f'(\varphi(x))} = \delta_{\mathscr{H}}(\varphi(x)).$$

This shows that φ is a δ-homomorphism. ∎

PROPOSITION 1.3. *Let $\mathscr{A}$ be an integrally closed δ-domain, let $\mathscr{K}$ be its field of fractions, and let $\mathscr{L}$ be a separable normal algebraic extension of $\mathscr{K}$. Let $\mathscr{C}$ be the integral closure of $\mathscr{A}$ in $\mathscr{L}$. Then every prime ideal of $\mathscr{C}$ lying over a prime δ-ideal of $\mathscr{A}$ is a δ-ideal.*

Proof. Let $\mathfrak{q}$ be a prime ideal in $\mathscr{C}$ such that $\mathfrak{p} = \mathfrak{q} \cap \mathscr{A}$ is a prime δ-ideal of $\mathscr{A}$. By Proposition 1.1, there is a prime δ-ideal $\mathfrak{q}'$ of $\mathscr{C}$ lying over $\mathfrak{p}$. By Theorem 9.3 in [Ma], there is σ, a field automorphism of $\mathscr{L}$ over $\mathscr{K}$ such that $\sigma(\mathfrak{q}') = \mathfrak{q}$. Now it follows from Lemma 1.2 (setting $\mathscr{H}$ equal to $\mathscr{L}$) that σ is in fact a differential field automorphism; hence, $\mathfrak{q}$ is a δ-ideal as well. ∎

The next result shows that the assumptions for the going-down theorem are enough to ensure that every prime ideal lying over a prime δ-ideal is differential.

THEOREM 1.4. *Let $\mathscr{A} \subseteq \mathscr{B}$ be δ-domains with $\mathscr{B}$ integral over $\mathscr{A}$ and $\mathscr{A}$ integrally closed. Then every prime ideal lying over a prime δ-ideal of $\mathscr{A}$ is a δ-ideal.*

Proof. Let $\mathscr{K} = \langle \mathscr{A} \rangle$ be the field of fractions of $\mathscr{A}$. Since $\mathscr{B}$ is integral over $\mathscr{A}$, $\langle \mathscr{B} \rangle$ is an algebraic extension of $\mathscr{K}$. Let $\mathscr{L}$ be the normal closure of $\langle \mathscr{B} \rangle$ in the algebraic closure of $\mathscr{K}$. By the assumption $\mathbb{Q} \subseteq \mathscr{A}$, $\operatorname{Char} \mathscr{K} = 0$, and hence $\mathscr{L}/\mathscr{K}$ is separable. Let $\mathscr{C}$ be the integral closure of $\mathscr{A}$ in $\mathscr{L}$. Since $\mathscr{B}$ is integral over $\mathscr{A}$, $\mathscr{C}$ is also the integral closure of $\mathscr{B}$ in $\mathscr{L}$. Let $\mathfrak{q}$ be a prime ideal of $\mathscr{B}$ such that $\mathfrak{p} = \mathfrak{q} \cap \mathscr{A}$ is a prime δ-ideal of $\mathscr{A}$. Since $\mathscr{C}$ is integral over $\mathscr{B}$, by the usual lying over theorem, there is a prime ideal $\mathfrak{r}$ lying over $\mathfrak{q}$. So $\mathfrak{r}$ is lying over $\mathfrak{p}$; thus it follows from Proposition 1.3 that $\mathfrak{r}$ is a δ-ideal and so is $\mathfrak{q} = \mathscr{B} \cap \mathfrak{r}$. ∎

Let $\mathscr{F}$ be a differentially closed field of characteristic 0. We will identify δ-$\mathscr{F}$-varieties with their $\mathscr{F}$-points. Let X be a δ-$\mathscr{F}$-variety with $\operatorname{td}_{\delta}(X) = d$; a chain of δ-subvarieties in X is called a **long chain** if its length is greater than or equal to ωd. We will show that, for algebraic varieties, there is a long chain starting at any given point.

Remark 1.5. Before restricting ourselves to the case where X is an algebraic variety, let us point out the following:

(1) This is a local problem: Let x be any point in X and let U be a δ-open neighborhood of x. If we can find a long chain

$$x = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{\omega d}$$

in U, then the δ-closures of the Z_αs in X will form a long chain in X starting at x. So we can always assume that X is affine.

(2) In fact, one can go the other way: Suppose $Z_0 \subsetneq Z_1$ are δ-subvarieties of X and U is a δ-open subset such that $U \cap Z_0$ is nonempty. Since the Z_is are δ-irreducible, they are equal to the δ-closure of their intersection with U. So we must have $U \cap Z_0 \subsetneq U \cap Z_1$. Therefore, if $\boldsymbol{C}$ is any infinite increasing chain of δ-subvarieties in X, then the chain in U obtained by intersection will have the same length as $\boldsymbol{C}$ provided that some finite term of $\boldsymbol{C}$ has a nonempty intersection with U.

(3) The Kolchin Catenary Problem is "generically" solved: For any δ-$\mathcal{F}$-variety X, there is a proper δ-closed subset Y of X such that, for any $x \in X \setminus Y$, there is a long chain starting at x. This is proved by Rosenfeld in [Ro, Section III]. Also he dealt with chains starting from δ-subvarieties of arbitrary δ-transcendence degree instead of just points.

So the essence of Kolchin's problem is: what happens at the "singular" points? Since one can resolve singularities for algebraic varieties in characteristic 0, we are able to answer the question in that case.

2. THE SMOOTH CASE

We know that a long chain exists at every point of $\mathbb{A}^d$. For example, the chain of δ-prime ideals

$$\begin{aligned}[y_1, y_2, \ldots, y_d] &\supset [\delta y_1, y_2, \ldots, y_d] \supset [\delta^2 y_1, y_2, \ldots, y_d] \\ &\supset \cdots \supset [y_2, \ldots, y_d] \supset [\delta y_2, \ldots, y_d] \supset \cdots \supset 0\end{aligned}$$

corresponds to a long chain, $\boldsymbol{S}$, starting at the origin. In fact, we know a little bit more:

PROPOSITION 2.1. *Given U a Zariski open subset of $\mathbb{A}^d$, there is a long chain in $\mathbb{A}^d$ starting at the origin such that its intersection with U is a long chain in U.*

Proof. By Remark 1.5(2), $\boldsymbol{S} \cap U$ will be a long chain in U if one of the finite terms of $\boldsymbol{S}$ has a nonempty intersection with U. Let $y_1, \dots, y_d$ be the coordinates of $\mathbb{A}^d$. Certainly U may not contain $S_0 = \{0\}$. However, by a linear transform of $\mathbb{A}^d$, say g, we can make sure that $g(\mathbb{A}^d \backslash U)$ intersects the y_1 axis in a proper Zariski closed subset (hence finite). So S_1 as an infinite subset of the y_1 axis intersects $g(U)$. It follows that $g^{-1}(\boldsymbol{S})$ is a chain satisfying the requirements given in the proposition. Finally, it is evident that if $0 \in U$, then $g^{-1}(\boldsymbol{S}) \cap U$ is a long chain in U starting at 0. ∎

Let X be an $\mathcal{F}$-variety of dimension d; by Noether normalization, there is a finite surjection from X to $\mathbb{A}^d$. To get a long chain at a given point in X, the idea is to apply the differential version of the going-down theorem to a long chain in $\mathbb{A}^d$. As pointed out earlier, we can assume $X = \operatorname{Spec} B$ to be affine. The finite surjection from X to $\mathbb{A}^d$ corresponds to an embedding of $A = \mathcal{F}[y_1, \dots, y_d]$ into B which makes B into a finite A-module. In the light of Proposition 1.1, the arguments above work as long as the property of "being finite" can be prolongated, i.e., if $B_\infty = \mathcal{F}\{X\}$ is a finite $A_\infty = \mathcal{F}\{y_1, \dots, y_d\}$ module. However, one may run into trouble at the very first step of the prolongation. For example, take $B = \mathcal{F}[x, y]/(y^2 - x^3 + 1)$ and $A = \mathcal{F}[x]$. It is easy to see that the image of δy in the first prolongation

$$B_1 = \frac{\mathcal{F}[x, \delta x, y, \delta y]}{(y^2 - x^3 + 1, 2y\delta y - 3x^2\delta x)}$$

is not integral over $A_1 = \mathcal{F}[x, \delta x]$. Hence B_1 is not a finite A_1-module. However, note that in this example the problem happens exactly at the branch points of the map $\operatorname{Spec} \mathcal{B} \to \operatorname{Spec} \mathcal{A}$. In fact, one can easily check that δy is integral over x and δx once we localize to the open set $x^3 - 1 \neq 0$.

We will start with the case where there are no branch points and make the arguments above precise. The following result of Buium [Bu, Chapter 4, Proposition 4.6] describes the canonical prolongation of an étale covering between smooth $\mathcal{F}$-varieties. Recall that $\mathcal{O}_X^{[r]}$ denotes the sheaf of δ-polynomial function of order $\leq r$ on an $\mathcal{F}$-variety X.

PROPOSITION 2.2 (Buium). *Let u: $X \to Y$ be an étale finite covering of smooth $\mathcal{F}$-varieties and let u^r: $X^r \to Y^r$ be the induced morphisms between the canonical infinite prolongation sequences. Then the squares*

$$\begin{array}{ccc} X & \longleftarrow & X^r \\ \big\downarrow u & & \big\downarrow u^r \\ Y & \longleftarrow & Y^r \end{array}$$

are Cartesian. Consequently, if u is a Galois covering with group G, then

$$\mathcal{O}^{[r]}(Y) = \mathcal{O}^{[r]}(X)^G, \qquad r \geq 0.$$

LEMMA 2.3. *Let $X = \operatorname{Spec} A$ be a smooth affine $\mathcal{F}$-variety. Then A_∞, the canonical infinite prolongation of A, is integrally closed.*

Proof. Let $X^r = \operatorname{Spec} A_r$ be the rth term in the canonical infinite prolongation of $X \to \operatorname{Spec} \mathcal{F}$. By [Bu, Chapter 3, 2.5], all the X^r are smooth affine varieties. In particular the A_rs are integrally closed. Also in the smooth case, the map $X^{r+1} \to X^r$ will have a torsor structure under the relative tangent bundle $V(T_{X^r/X^{r-1}})$. Therefore, $X^{r+1} \to X^r$ will be surjective, and hence the induced map $A_r \to A_{r+1}$ is injective. Let K_r $(r \le \infty)$ be the field of fractions of A_r. Identifying A_r as a subring of A_{r+1}, we have $A_\infty \cong \cup A_r$. This also induces an isomorphism between K_∞ and $\cup K_r$. Now the fact that A_∞ is integrally closed follows from the fact that each A_r is integrally closed. ∎

PROPOSITION 2.4. *Let $f: X \to Y$ be an étale finite covering of affine smooth $\mathcal{F}$-varieties. If $W_0 \subset W_1 \subset \cdots \subset W_\alpha$ is an increasing chain of δ-subvarieties of Y and V_0 is a δ-subvariety of X such that $f(V_0) = W_0$, then there exists an increasing chain of δ-subvarieties of X, $V_0 \subset V_1 \subset \cdots \subset V_\alpha$, such that $f(V_\beta) = W_\beta$ for every $\beta \le \alpha$.*

Proof. Let $X = \operatorname{Spec} B$, let $Y = \operatorname{Spec} A$, and let $u: A \to B$ be the induced ring homomorphism. Since f is a finite surjection, u is injective and B is a finite A-module. Let $B = As_1 + As_2 + \cdots + As_n$, with $s_1 = 1 \in B$. Let A_r and B_r be the rth canonical prolongation of A and B, respectively. Then by Proposition 2.2, $B_r = B \otimes_A A_r = \sum_{i=1}^n A_r s_i$ for each r. So B_∞ is generated as an A_∞-module by the images of the s_is. In particular, B_∞ is still integral over A_∞. Let $\mathfrak{q}_0$ and $\mathfrak{p}_\beta$ $(\beta \le \alpha)$ be prime δ-ideals in B_∞ and A_∞, respectively, corresponding to V_0 and W_β. Since $f(V_0) = W_0$, we have $\mathfrak{q}_0$ lying over $\mathfrak{p}_0$. By Lemma 2.3, A_∞ is integrally closed, so we can apply Proposition 1.1 successively to obtain a decreasing chain $\{\mathfrak{q}_\beta\}_{\beta \le \alpha}$ of prime δ-ideals of B_∞. Note that if $\gamma \le \alpha$ is a limit ordinal, then $\mathfrak{q}' := \cap_{\beta<\gamma} \mathfrak{q}_\beta$ is a prime δ-ideal lying over $\mathfrak{p}' := \cap_{\beta<\gamma} \mathfrak{p}_\beta$. If $\mathfrak{p}_\gamma = \mathfrak{p}'$, then simply take $\mathfrak{q}_\gamma$ to be $\mathfrak{q}'$; otherwise, apply Proposition 1.1 to $\mathfrak{p}_\gamma \subsetneq \mathfrak{p}'$ to get $\mathfrak{q}_\gamma$. The δ-subvarieties V_βs corresponding to the $\mathfrak{q}_\beta$s form an increasing chain with $f(V_\beta) = W_\beta$ for all $\beta \le \alpha$. ∎

Our next goal is to prove the following statement.

Let X be an affine smooth $\mathcal{F}$-variety of dimension d. Then for any $x \in X$, there exists a finite surjection $\psi: X \to \mathbb{A}^d$ which is étale at some neighborhood of x.

Once we establish this, then by going to a smaller affine neighborhood of $\psi(x)$, we reduce ourselves to the situation described in Proposition 2.4. In fact, we will prove a slightly stronger result (see Proposition 2.7).

The next two propositions are auxiliary. The first one is a minor variation of a result in [Mu, Chapter 3, Sect. 7, Theorem 2].

PROPOSITION 2.5. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d and let x be a smooth point of X. Suppose H is a hyperplane in $\mathbb{P}^n$ and $x \notin H$. Then there exists a projection π: $\mathbb{P}^n \backslash M \to \mathbb{P}^{d+1}$ with center M, a linear subspace of H, such that π is an isomorphism near x.*

Proof. Suppose $n > d+1$; otherwise the statement is obvious. Following the arguments in [Mu], it suffices to find a linear subspace M of H such that:

(1) the join $J(M, x)$ intersects X at $\{x\}$ only.

(2) the projective tangent space, $T_{x,X}$, of X at x does not intersect M.

The fact that x is a smooth point implies $J(x, X)$, the joint of X and x, contains $T_{x,X}$ and $\dim J(x, X) = \dim(X) + 1 = d + 1$. Since x is not in H, the intersection of $J(x, X)$ and H is a proper Zariski closed subset of $J(x, X)$; hence it can have dimension at most d (in fact, it has dimension d). Thus one can find a linear subspace M of H of dimension $n-d-2$ such that M does not meet $J(x, X)$. So this M satisfies both (1) and (2). ∎

PROPOSITION 2.6. *Let X be a codimension-1 subvariety of $\mathbb{P}^{d+1}$. Suppose $x \in X$ is a smooth point and H is a hyperplane not containing x. Then there exists a projection φ: $\mathbb{P}^{d+1} \to \mathbb{P}^d$ with center at some point $p \in H$ such that $\varphi|_X$ is a finite surjection and it is étale over some affine neighborhood of $\varphi(x)$.*

Proof. Let $Bl_x(X)$ be the blow-up of X at x. Consider the map

$$\tilde{\pi}_x \colon Bl_x(X) \to H$$

induced by the projection π_x which sends $w \in X \backslash \{x\}$ to $\overline{xw} \cap H$. The branch locus of $\tilde{\pi}_x$ is a closed subset of codimension at least 1 in H. Since $X \cap H$ is a proper closed subset of X, by counting dimensions we have

$$d-1 \geq \dim(X \cap H) \geq d + d - (d+1) = d-1;$$

therefore, $\dim(X \cap H) = d-1$. Since x is a smooth point, T_x, the projective tangent space of X at x, is d-dimensional. By a similar argument, the dimension of $T_x \cap H$ is $d-1$. Thus we can find a point $p \in H$ avoiding the branch locus of $\tilde{\pi}_x$, $X \cap H$, and $T_x \cap H$ all together. Let φ be the projection with center at p; we argue that φ will satisfy the requirements.

Since $p \notin X$, by an argument in [Mu, Chapter 2, Sect. 7, Proposition 6], $\varphi|_X \colon X \to \mathbb{P}^d$ is a finite morphism. Moreover, X has codimension 1, so the map is surjective. For simplicity, we write φ instead of $\varphi|_X$. It remains to show that φ is étale over a neighborhood of $y := \varphi(x)$. First let us show that φ is unramified at every point of the fiber $\varphi^{-1}(y)$. The necessary and sufficient condition for $z \in X$ to be an unramified point of φ is $p \notin T_z$. Since we choose p away from T_x, this guarantees φ is unramified at x. Now if w is another point in the fiber, we have p, x, and w collinear. So w is a ramified point of $\varphi \iff p \in T_w \iff \overline{xw} = \overline{xwp} = \overline{wp} \subseteq T_w$. Note that the condition $\overline{xw} \subseteq T_w$ is also equivalent to that w is a ramification point of $\tilde{\pi}_x$. But then $p\,(=\overline{xw} \cap H) \in \{\text{branch locus of } \tilde{\pi}_x\}$. This contradicts our choice of p. Now it follows that none of the points in the fiber over y is singular; indeed, if $z \in \varphi^{-1}(y)$ is a singular point, then $\dim \Omega_{X,z} = d+1$, so the map of sheaves $\varphi^*\Omega_{\varphi(X)} \to \Omega_X$ could not be surjective at z, which contradicts the fact that φ is unramified at z. Since $\mathbb{P}^d$ is normal and φ is a finite map, it is an open map [Ma, Theorem 9.6]. So by going to a smaller affine neighborhood of y, we can assume φ is an unramified finite surjective map between nonsingular varieties. In particular, $\mathfrak{m}_x = \varphi^*(\mathfrak{m}_{\varphi(x)})\mathcal{O}_x$ for all x in the domain. By a theorem in [Mu, Chapter 3, Sect. 5, Theorem 4], we have φ is an étale map. ∎

Combining both Propositions 2.5 and 2.6, we get

PROPOSITION 2.7. *Let X be a smooth affine variety of dimension d. For any $x \in X$, there exists ψ: $X \to \mathbb{A}^d$, which is a finite surjection and is étale over some affine neighborhood of $\psi(x)$.*

Proof. Embed X into $\mathbb{P}^n_{\bar{x}}$ (for some $n \geq d+1$) via $X \subset \mathbb{A}^n = \mathbb{P}^n \backslash H$, where $H = \{x_0 = 0\}$. Let $\overline{X}$ be the Zariski closure of X in $\mathbb{P}^n$. Apply Proposition 2.5 to $\overline{X}$ and H to get a projection π: $\mathbb{P}^n \to \mathbb{P}^{d+1}$ centered at M, a linear subspace of H, and a neighborhood W of x such that $\pi|_W$ is an isomorphism between W and $V := \pi(W)$. Note that $\pi(H\backslash M)$ is a hyperplane in $\mathbb{P}^{d+1}$ which does not contain $\pi(x)$, and $\pi(\overline{X})$ is of codimension 1 in $\mathbb{P}^{d+1}_{\bar{y}}$. So by Proposition 2.6, there exists some $p \in \pi(H\backslash M)$ and projection φ at p such that $\varphi|_{\pi(\overline{X})}$ is a finite surjection which is étale over some neighborhood U of $\varphi\pi(x)$. Since p belongs to $\pi(H\backslash M)$, $\varphi \circ \pi$ is a projection with center L, an $(n-d-1)$-dimensional subspace of H. Denote this composition by π_L. By an automorphism of $\mathbb{P}^d$, if necessary, we can assume π_L is given by the linear forms $l_0 = x_0, l_1, \ldots, l_d$. In this case, if we set $V_0 = \mathbb{P}^d \backslash \{y_0 = 0\}$, then $\pi_L^{-1}(V_0) \cap \overline{X} = \overline{X}\backslash H = X$. Therefore $\psi := \pi_L|_X$ is a finite map from X onto $V_0 \cong \mathbb{A}^d$. Since étale maps between varieties are open, we see that ψ is étale over some open neighborhood of $\psi(x)$ [e.g., one can take $\varphi(\varphi^{-1}(U) \cap V) \cap V_0$]. ∎

The next result settles the question when X is a smooth variety.

THEOREM 2.8. *Let X be a smooth $\mathcal{F}$-variety of dimension d. Then for any $x \in X$, there is a long chain starting at x.*

Proof. As we have pointed out in Remark 1.5, one can assume X is affine to begin with. By Proposition 2.7, there is a finite surjection $f\colon X \to \mathbb{A}^d$ which is étale over an affine neighborhood U of $f(x)$. Without loss of generality, we can assume $f(x) = 0$, and we have the following situation

$$\begin{tikzcd} x \in V \arrow[r] \arrow[d, "f|_V"] & X \arrow[d] \\ 0 \in U \arrow[r] & \mathbb{A}^d \end{tikzcd},$$

where $V = f^{-1}(U)$. Since étale maps are affine, $f|_V$ is an étale finite covering between smooth affine varieties. By Proposition 2.1, we get a long chain in U starting at 0. Now the theorem follows from Proposition 2.4. ∎

3. THE GENERAL CASE

In order to prove the result in the singular case, we need to sharpen Theorem 2.8 slightly.

PROPOSITION 3.1. *Let X be a smooth $\mathcal{F}$-variety of dimension d, let $x \in X$, and let E be a codimension-1 closed subset of X containing x. Then there exists a long chain $\{W_\alpha\}$ in X starting at x such that $W_\beta \backslash E \subsetneq W_\gamma \backslash E$, for all $\beta < \gamma$.*

Proof. We keep the same assumptions and notation as in Theorem 2.8. Let H be a codimension-1 closed subset of $\mathbb{A}^d$ containing $f(E)$. Applying Proposition 2.1 to the open set $U \backslash H$, we get a long chain $\{Z_\alpha\}$ in $\mathbb{A}^d$ starting at 0 such that $Z_\beta \backslash H \subsetneq Z_\gamma \backslash H$, for all $\beta < \gamma$. By Proposition 2.4, there is a long chain $\boldsymbol{C}$ in V lying over $\{Z_\alpha \cap U\}$. The chain $\{W_\alpha\}$ obtained by taking δ-closure in X of each term of $\boldsymbol{C}$ has the required properties. ∎

THEOREM 3.2. *The smoothness assumption in Theorem 2.8 is unnecessary.*

Proof. As before, we may well assume X is affine and x is contained in every singular component of X. Let Z be the union of all the singular components. By resolution of singularities in characteristic 0, we have a smooth model $\pi\colon \widetilde{X} \to X$. Let $E = \pi^{-1}(Z)$ be the exceptional divisor and let $U = X \backslash Z$; then π induces an isomorphism from $\widetilde{X} \backslash E$ to U. Pick any $\tilde{x} \in \pi^{-1}\{x\}$. By Proposition 3.1, we get a long chain $\{W_\alpha\}$ starting at $\tilde{x}$ such that $W_\beta \backslash E \subsetneq W_\gamma \backslash E$, for all $\beta < \gamma$. The δ-closures of the $\pi(W_\alpha)$s will form a long chain of X starting at x. ∎

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